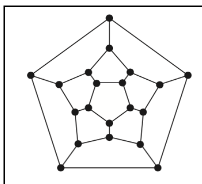
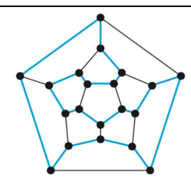
Discrete Structures (Mathematics)
Final Term Examination (Duration: 2Hours)

Total Marks: 60	Course Code: CMP-2111

Note: All questions carry equal marks.

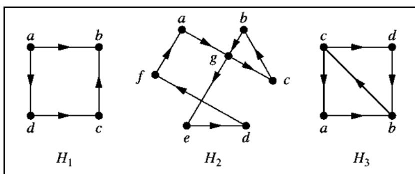
Q1: Is there a Hamilton circuit in the graph shown in the following figure? (5)





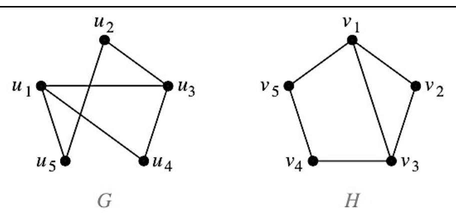
Solution:

Q2: Which of the digraphs in the following figure have an Euler circuit? Of those that do not, which have an Euler path? (5)



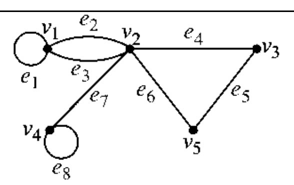
Solution: The graph H_2 has an Euler circuit, for example, a, g, c, b, g, e, d, f, a. Neither H_1 nor H_3 has an Euler circuit (as the reader should verify). H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not.

Q3: Determine whether the graphs G and H shown in the following figure are isomorphic. (5)



Solution: Both G and H have five vertices and six edges, both have two vertices of degree three and three vertices of degree two, and both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five. Because all these isomorphic invariants agree, G and H may be isomorphic.

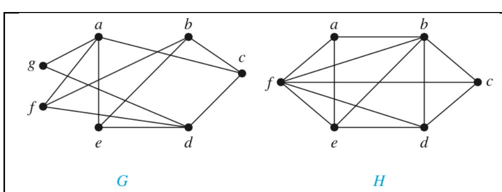
Q4: Represent the following graph using an incidence matrix. (5)



Solution:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Q5: Are the graphs G and H displayed in the following figure bipartite? (5)



Solution: Graph G is bipartite because its vertex set is the union of two disjoint sets, {a, b, d} and {c, e, f, g}, and each edge connects a vertex in one of these subsets to a vertex in the other subset. (Note that for G to be bipartite it is not necessary that every vertex in {a, b, d} be adjacent to every vertex in {c, e, f, g}. For instance, b and g are not adjacent.)

	Graph H is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset. (The reader should verify this by considering the vertices a, b, and f .)

Q6: Prove that An undirected graph has an even number of vertices of odd degree. (5)

Proof: Let V1 and V2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph G = (V , E) with m edges. Then $2m = \sum_{v \in V} \deg(v) = \sum_{v \in V1} \deg(v) + \sum_{v \in V2} \deg(v)$. Because deg(v) is even for v ∈ V1, the first term in the right-hand side of the last equality is even. Furthermore, the sum of the two terms on the right-hand side of the last equality is even, because this sum is 2m. Hence, the second term in the sum is also even. Because all the terms in this sum are odd, there must be an even number of such terms. Thus, there are an even number of vertices of odd degree.

Q7: Define the HANDSHAKING and FERMAT'S LAST theorems. (5)

Solution: THE HANDSHAKING THEOREM Let G = (V , E) be an undirected graph with m edges. Then $2m = \sum_{v \in V} \deg(v)$. (Note that this applies even if multiple edges and loops are present.). FERMAT'S LAST THEOREM The equation $x^n + y^n = z^n$ has no solutions in integers x, y, and z with $xyz \neq 0$ whenever n is an integer with n > 2.

Q8: Find the join and meet of the zero–one matrices given below. (5)

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	*Solution:* We find that the join of **A** and **B** is $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The meet of **A** and **B** is $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Q9: Show that the set of real numbers is an uncountable set. (5)

Solution: See the lecture slides

Q10: Let x be a real number with |x| <1. Find $\sum_{n=0}^{\infty} x^n$ (5)

Solution: By Theorem 1 with $a = 1$ and $r = x$ we see that $\sum_{n=0}^{k} x^n = \frac{x^{k+1} - 1}{x - 1}$. Because $|x| < 1$, x^{k+1} approaches 0 as k approaches infinity. It follows that

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = \frac{0 - 1}{x - 1} = \frac{1}{1 - x}.$$

We can produce new summation formulae by differentiating or integrating existing formulae.

Q11: Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047. (5)

Solution: See in the lecture slides

Q12: Prove or disprove that ⌈x + y⌉ = ⌈x⌉ + ⌈y⌉ for all real numbers x and y. (5)

Solution: Although this statement may appear reasonable, it is false. A counterexample is supplied by $x = \frac{1}{2}$ and $y = \frac{1}{2}$. With these values we find that $\lceil x + y \rceil = \lceil \frac{1}{2} + \frac{1}{2} \rceil = \lceil 1 \rceil = 1$, but $\lceil x \rceil + \lceil y \rceil = \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil = 1 + 1 = 2$.